Hydrophones Mathematical Model

Colin Bradbury

This note describes the high-level mathematical model used to determine the location of the pinger using the differences in reception time of the emitted signal at the various hydrophones. The method for detecting the signal and determining the time differences is (or will be) described elsewhere. The objective of this mathematical model is to determine the bearing of the pinger relative to the submarine. The distance to the pinger and the relative depth of the pinger are of no concern in this model.

Other researchers have used a sub-centric model which puts the submarine in a fixed position and then attempts to locate the pinger relative to the submarine. This method uses a pinger-centric approach which puts the pinger in a fixed position and then determines the attitude of the sub with respect to the pinger. Note that in the real world, the pinger doesn't move but the submarine does.

Submarine Definitions

If any pair hydrophones receive the pinger signal simultaneously, the pinger is located in the plane perpendicular to the line connecting the hydrophones – declare victory and go home. Otherwise, let H1 denote the first hydrophone to receive the pinger signal, H3 denote the last hydrophone to receive the pinger signal, and H2 denote the next-to-last hydrophone to receive the pinger signal. Thus the pinger signal arrives at H1 then H2 then H3. Any other hydrophones are of no concern in this model.

Let the distances between the hydrophones be denoted by A, B, and C where A is the length of the side opposite H1, B is the length of the side opposite H2 and C is the length of the side opposite H3.

Time Differences

Let T12 be the time difference between the arrival times of the pinger signal at hydrophones H1 and H2 respectively. Similarly, let T13 be the time difference between the arrival times of the pinger signal at hydrophones H1 and H3 respectively. Let d12 be the distance that the pinger signal travels in time T12, and let d13 be the distance that the pinger signal travels in time T13. The distances d12 and d13 can be calculated by multiplying the time differences T12 and T13 respectively by the speed of sound in water (approximately 1482 meters/second, depending on temperature and salinity).

Coordinate System

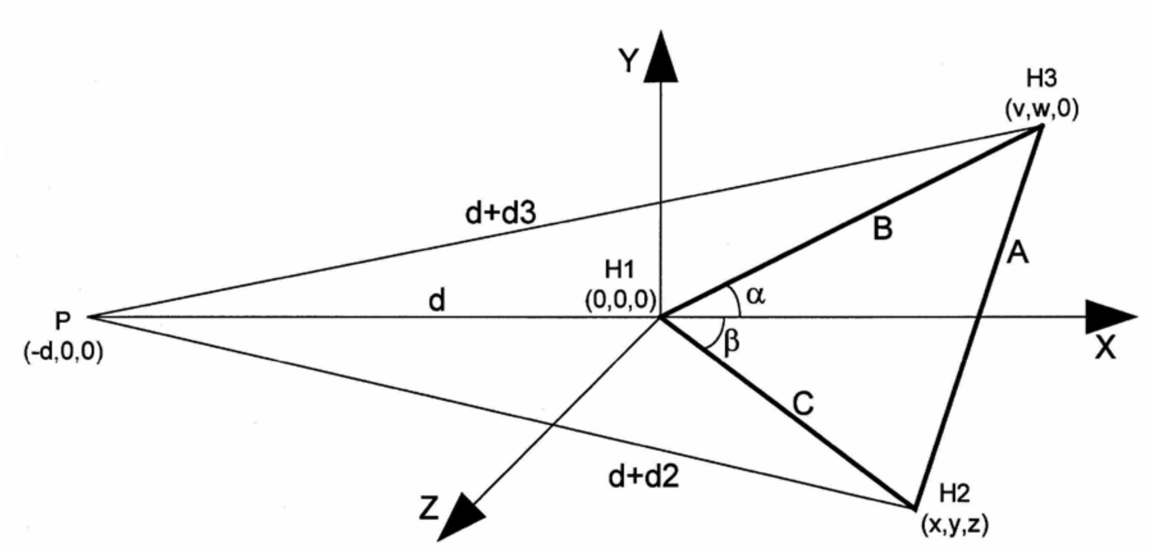
Construct a Cartesian coordinate system as follows: Let the coordinate origin be at hydrophone H1. Let the X axis be the line passing through the pinger and H1 such that the pinger P is a distance d from H1 in the negative direction (i.e. P is at (-d, 0, 0)).

Let the Y axis be the line perpendicular to the X axis such that hydrophone H3 lies in the X-Y plane at (v, w, 0) where w is a non-negative real value and the distance between H3 and the pinger P is d+d13. Let Alpha (α) be the angle between the positive X axis and the line connecting H1 and H3.

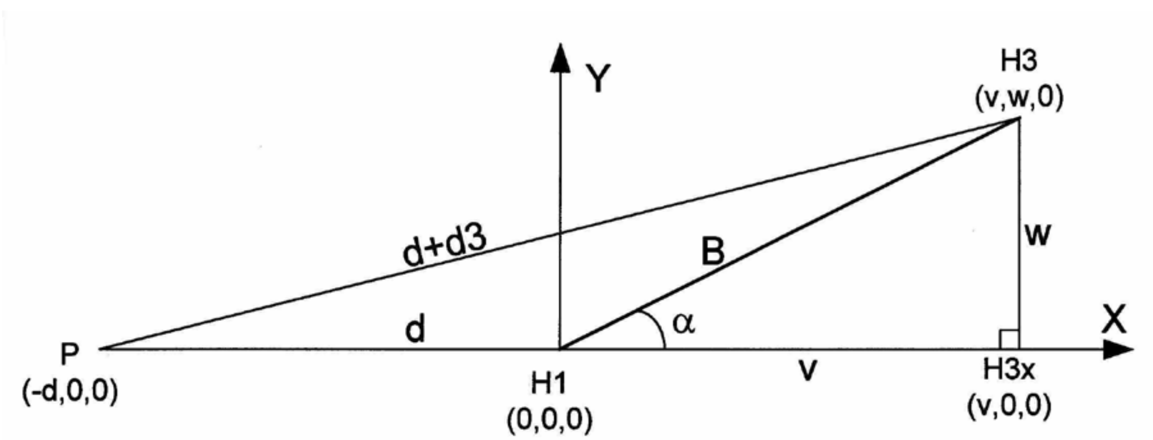
Let the Z axis be the line perpendicular to the X and Y axes such that hydrophone H2 lies at coordinates (x, y, z) where z is a non-negative real value and the distance between H2 and the pinger P is d+d12. Let Beta (β) be the angle between the positive X axis and the line connecting H1 and H2.

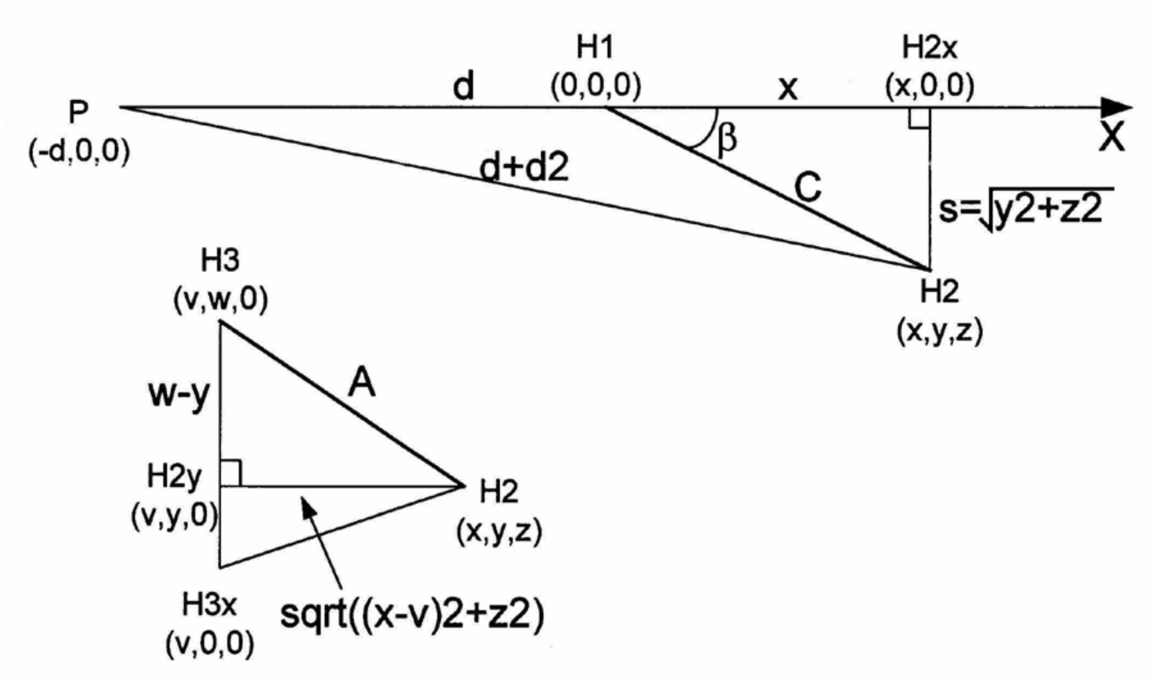
Mathematical Model

The sum of the above definitions gives the following representation of the model:



For ease in understanding the model, it can be deconstructed into three planar views: (1) the X-Y plane alone, (2) the oblique plane containing the X axis and the hydrophone H2, and (3) the oblique plane containing hydrophones H2 and H3 plus the projection of H3 onto the X axis. These planar views appear as follows:





In the last oblique plane, point H3x at coordinates (v, 0, 0) is the projection of H3 onto the X axis, and the point H2y at coordinates (v, y, 0) is the projection of H2 onto the line between H3 and H3x. Both of these projections meet the line they are projected onto at right angles as shown.

Analysis of the triangles in the three selected planes gives the following equations:

$$v = B \cos(\alpha) \qquad w = B \sin(\alpha)$$

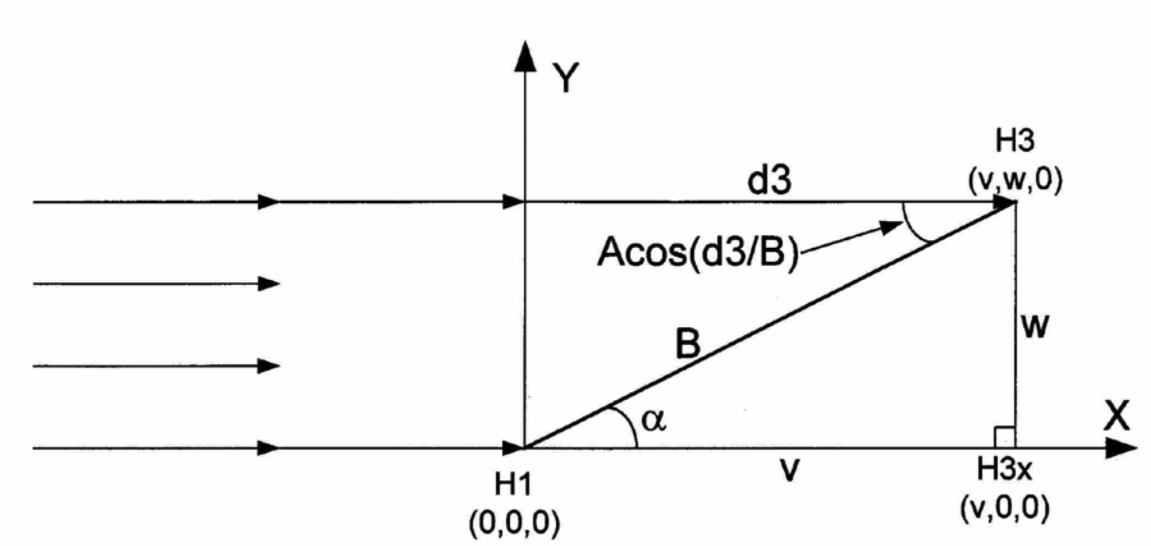
$$x = C \cos(\beta) \qquad y^2 + z^2 = C^2 \sin^2(\beta)$$

$$A^2 = (w - y)^2 + (x - v)^2 + z^2$$

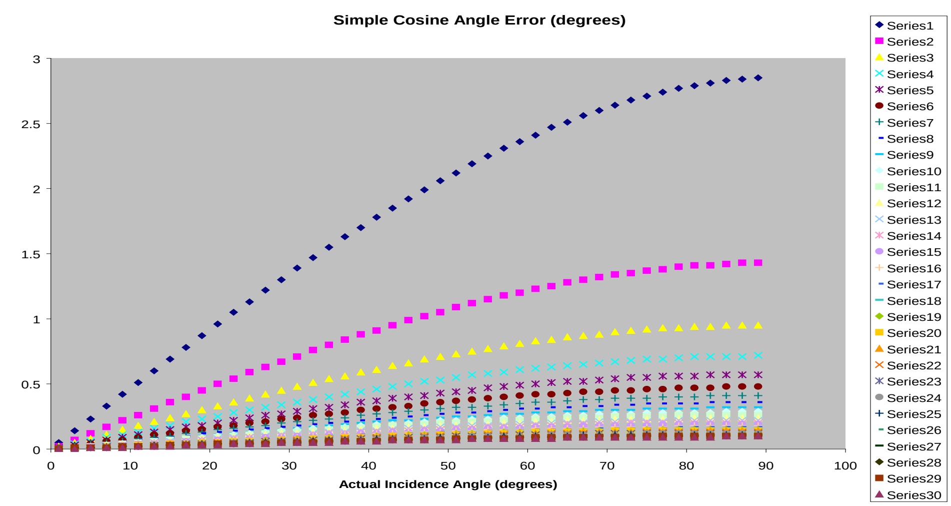
Clearly, these equations are easy to solve once the angles Alpha and Beta are known. A simple coordinate transformation can then convert the results into sub-centric form and out pops the bearing of the pinger. The real problem then is just one of determining the incidence angles Alpha and Beta.

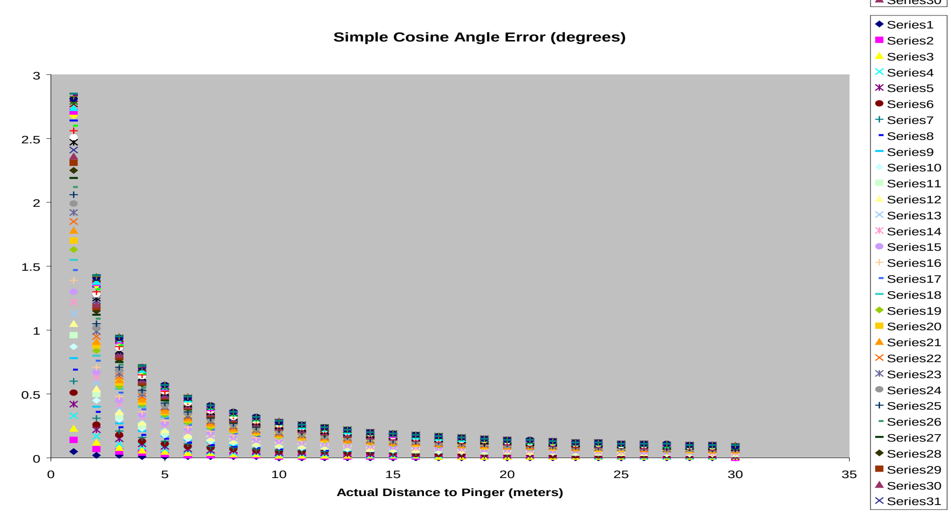
Incidence Angles

Previously, researchers have reasoned that with a pinger sufficiently far away, the signal paths to the different hydrophones are approximately parallel. This assumption leads to the conclusion that the length difference (derived from the time difference for the arrival times of the pinger signal) divided by the distance between the hydrophones gives the cosine of the Incidence Angle. Graphically, this is illustrated as follows:

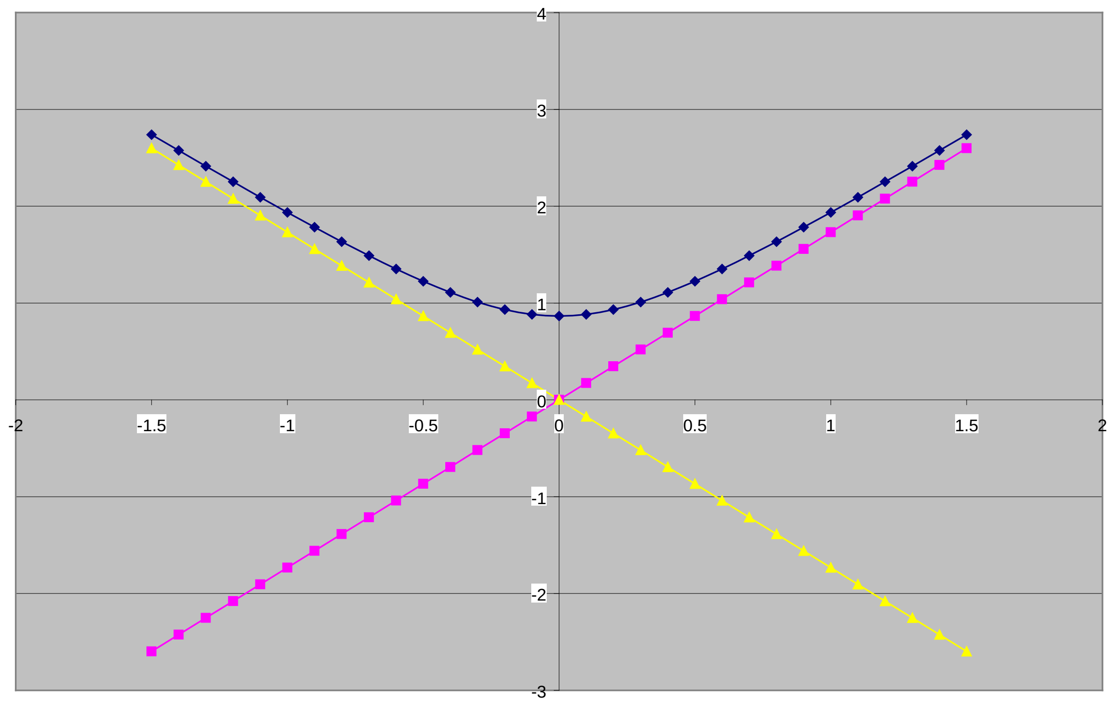


The graphs of the error factors in the simple cosine method show that it always underestimates the incidence angle and the problem gets worse as the submarine approaches the pinger.



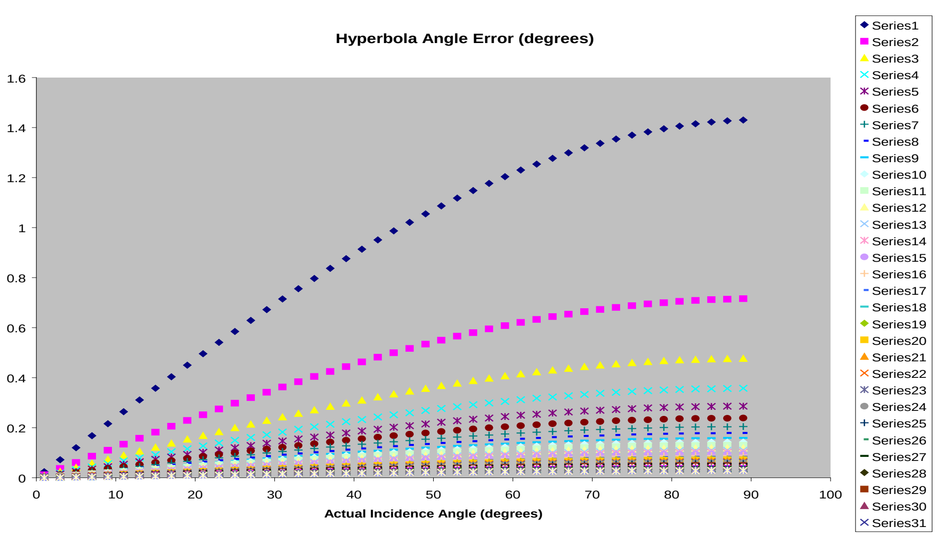


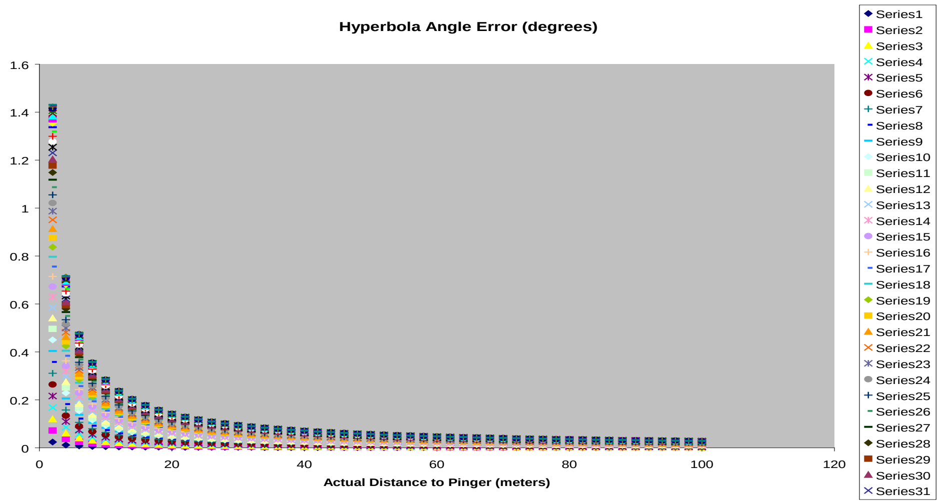
Mathematically, the difference in the lengths to two known points being a constant defines a hyperbola with the two points (the hydrophones) at the foci. The intercept point where the hyperbola crosses the line between the two hydrophones is easy to calculate and so the equation of the hyperbola is known. As the distance of the point from the foci increases, the point approaches the hyperbola asymptote lines. These lines cross at the midpoint between the hydrophones and have known equations derived from the hyperbola equation. Graphically, this is illustrated as follows:



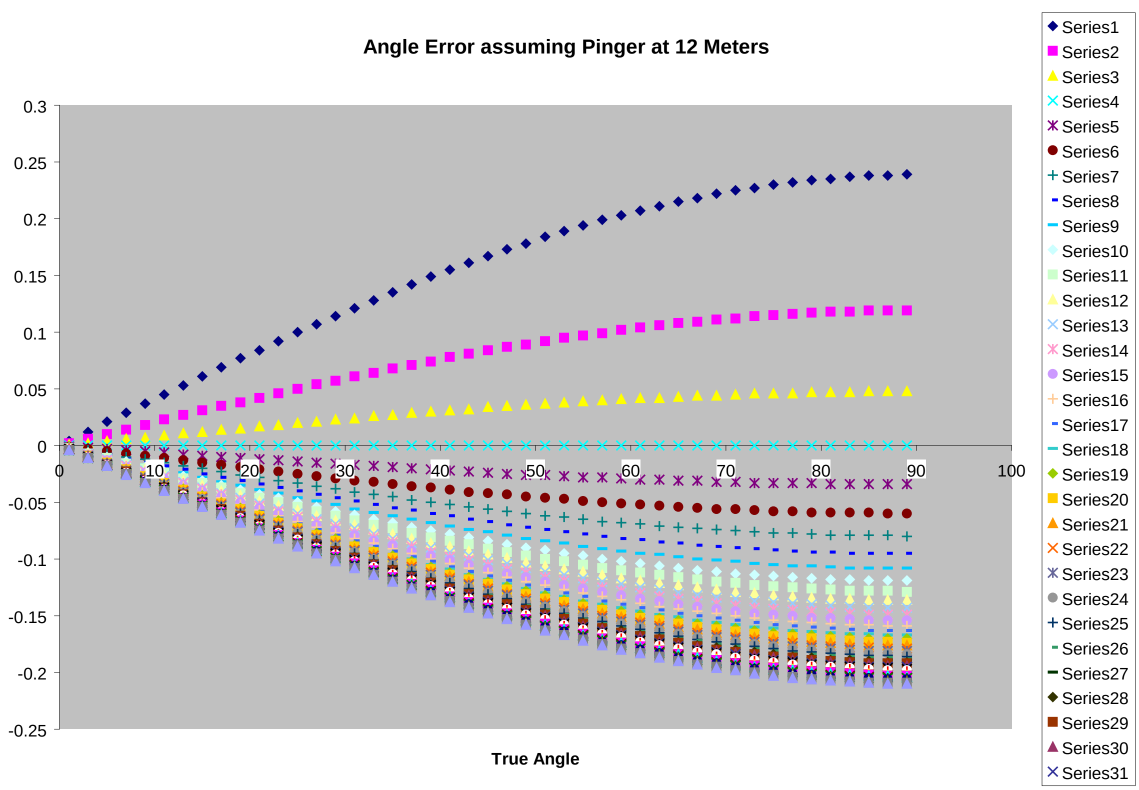
Hyperbola and Asymptotes

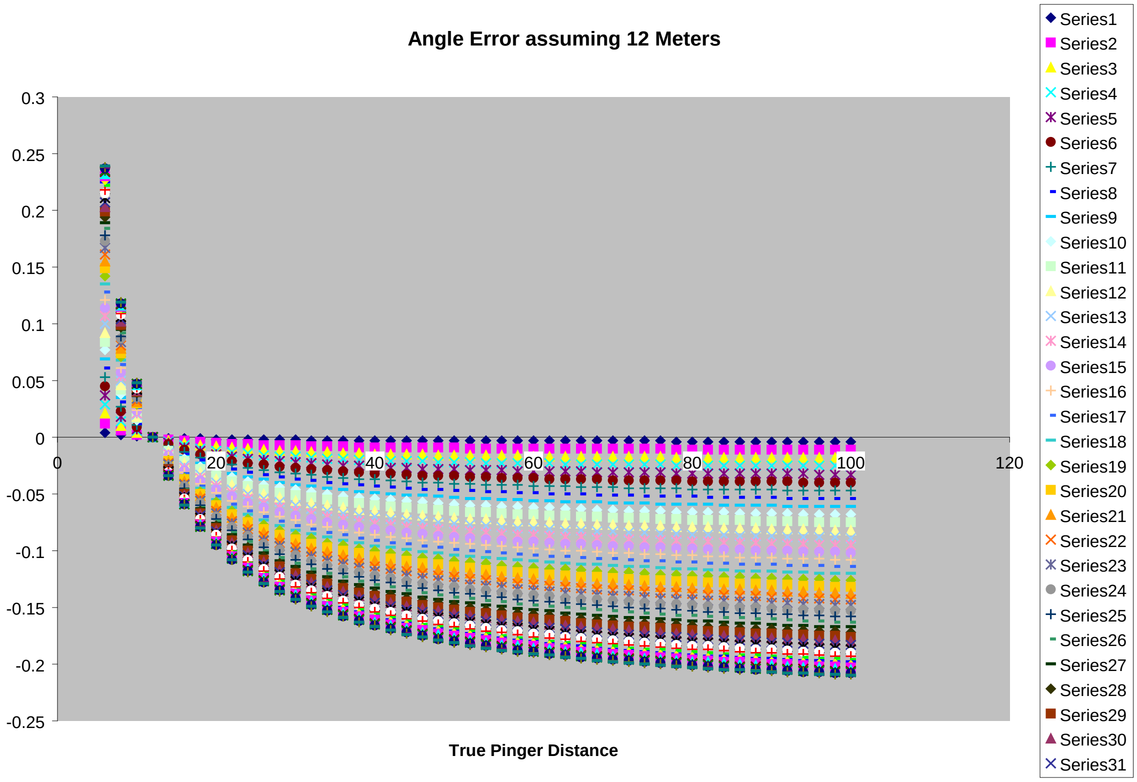
The graphs of the error factors in the hyperbola method show that it does better than the simple cosine method but it too always underestimates the incidence angle and the problem gets worse as the submarine approaches the pinger.





Referring back to the model pictures, it is appears that when the incidence angle is known, the distance to the pinger can be calculated. The reverse is also true. If the distance to the pinger is known then the incidence angle can be calculated. The assumption that the pinger is at a preset distance from the submarine gives another estimate of the incidence angle. If the assumed distance is further than the actual distance, the angle estimate is too low, but if the actual distance is less, the angle estimate is too high. In competition, the distance to the pinger is limited by the size of the competition arena. This makes it possible to choose a preset distance such that the angle error is equally distributed about zero. Note that the angle error when the pinger is actually at the preset distance is zero.





The natural follow-on to the last thought is that any of the above methods can be used to get an initial estimate of the distance to the pinger and then that estimate can be used in the preset-distance formula to get more accurate calculations. This turns out to be a bad idea. Trying to use any of the above methods to determine the distance to the pinger runs headlong into the computational nightmare called Massive Loss of Precision. This happens when trying to subtract two numbers that are very close together and then use the result for something. In this model, the distance d13 is really close to the distance v and the distance d12 is really close to the distance x. Doing either subtraction gives a number which is smaller than the error involved in computing d12 and d13 in the first place – the result is meaningless. The only way to increase the precision of the calculations is to increase the distance between the hydrophones. **This leads to the conclusion that the distance to the pinger cannot be calculated accurately by any sub in the competition**.

Fortunately, knowing the distance to the pinger isn't really useful anyway. The only figure of real interest is the bearing of the pinger relative to the sub. Starting over with this revised objective led to the Paradigm Shift. A paradigm shift is what happens when your subconscious suddenly jumps up, kicks you in the head and yells "Hey stupid, you're doing this all wrong." The algorithm to calculate only the bearing of the pinger is disgustingly simple. In fact, it's so disgusting that it's not fit for putting down on paper. The solution is left as an exercise for the reader.

This is the end of this White Paper. The use of these techniques without proper acknowledgement of the author in all written works will cause your sub to be cursed and sink to the bottom of the competition arena. Be sure to read the companion White Papers on the Model Simulation Program, Hydrophone Sampling Techniques, Pyramidal Frequency Search, and Synchronous Fourier Transforms. Coming soon to a website near you!