Synchronous Transform Mathematics

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The outcome of the synchronous transform data collection phase is a set of values (y_i) approximating to the sine waves of the pinger burst and a set of times (x_i) corresponding to when the sine waves were sampled ($0 <= x_i < 2\pi$). The equation of the sine waves is simply $f(x) = a \sin (x + b)$ where a and b are unknown constants, although $0 <= b < \pi/4$. The challenge is to find the best possible fit of the observed data to the theoretical equation. In particular, we wish to know the value of the constant b in order to determine the precise timing between the arrivals of the sine waves at different hydrophones. This is achieved by minimizing the value of

$$S = \sum_{i=1}^{N} (y_i - f(x_i))^2 = \sum_{i=1}^{N} (y_i - a \sin(x_i + b))^2 = \sum_{i=1}^{N} (y_i - a \sin(x_i)\cos(b) - a \cos(x_i)\sin(b))^2$$

The minimum value occurs when the first differential is zero, so differentiating with respect to the unknown variable a (and taking the summation limits as given) gives

$$\begin{aligned}
\delta S/\delta a &= 2 \sum \left[(y_i - a \sin(x_i)\cos(b) - a \cos(x_i)\sin(b)) \ (- \sin(x_i)\cos(b) - \cos(x_i)\sin(b)) \right] \\
&= -2 \sum \left[y_i \sin(x_i)\cos(b) + y_i \cos(x_i)\sin(b) - a \sin^2 (x_i)\cos^2 (b) - a \cos^2 (x_i)\sin^2 (b) \right. \\
&\qquad\qquad \left. - 2 a \sin(x_i)\cos(b)\cos(x_i)\sin(b) \right] \\
&= -2 \cos(b) \sum \left[y_i \sin(x_i) \right] -2 \sin(b) \sum \left[y_i \cos(x_i) \right] + 2 a \cos^2 (b) \sum \left[\sin^2 (x_i) \right] \\
&\qquad\qquad + 2 a \sin^2 (b) \sum \left[\cos^2 (x_i) \right] + 4 a \cos(b) \sin(b) \sum \left[\sin(x_i) \cos(x_i) \right] \\
&= -2 \cos(b)\, D -2 \sin(b)\, E + 2 a \cos^2 (b)\, F + 2 a \sin^2 (b)\, G + 4 a \cos(b) \sin(b)\, H
\end{aligned}$$

where

$D = \sum [y_i \sin(x_i)]$, $E = \sum [y_i \cos(x_i)]$, $F = \sum [\sin^2 (x_i)]$,

$G = \sum [\cos^2 (x_i)]$, and $H = \sum [\sin(x_i) \cos(x_i)]$, all known constants.

From this, it follows that

$$a = \frac{\cos(b)\, D + \sin(b)\, E}{\cos^2 (b)\, F + \sin^2 (b)\, G + 2 \cos(b) \sin(b)\, H}$$

Differentiating now with respect to the unknown variable b gives

$$\begin{aligned}
\delta S/\delta b &= 2 \sum \left[(y_i - a \sin(x_i)\cos(b) - a \cos(x_i)\sin(b)) \ (a \sin(x_i)\sin(b) - a \cos(x_i)\cos(b)) \right] \\
&= 2 \sum \left[a y_i \sin(x_i)\sin(b) + a y_i \cos(x_i)\cos(b) - a^2 \sin^2 (x_i)\cos(b)\sin(b) \right. \\
&\qquad \left. + a^2 \sin(x_i)\cos(x_i)\cos^2 (b) - a^2 \cos(x_i) \sin(x_i)\sin^2 (b) + a^2 \cos^2 (x_i)\sin(b) \cos(b) \right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\,a\sin(b) \sum [\, y_i \sin(x_i) \,] + 2\,a\cos(b) \sum [\, y_i \cos(x_i) \,] \\
&\qquad - a^2 \cos(b)\sin(b) \sum [\, \sin^2(x_i) \,] + a^2 \cos^2(b) \sum [\, \sin(x_i)\cos(x_i) \,] \\
&\qquad - a^2 \sin^2(b) \sum [\, \cos(x_i)\sin(x_i) \,] + a^2 \sin(b)\cos(b) \sum [\, \cos^2(x_i) \,] \\
&= 2\,a\sin(b)\,D + 2\,a\cos(b)\,E - a^2\cos(b)\sin(b)\,F + a^2\cos^2(b)\,H \\
&\qquad - a^2\sin^2(b)\,H + a^2\sin(b)\cos(b)\,G
\end{aligned}
$$

where D, E, F, G, and H are all as defined above.

From this, since $a > 0$, it follows that

$$
a = \frac{\sin(b)\,D + \cos(b)\,E}{\cos(b)\sin(b)\,F - \sin(b)\cos(b)\,G - \cos^2(b)\,H + \sin^2(b)\,H}
$$

Equating this to the previous result for the unknown variable a leads to the equation

$$
\begin{aligned}
&(\cos(b)\,D + \sin(b)\,E)\,(\cos(b)\sin(b)\,F - \sin(b)\cos(b)\,G - \cos^2(b)\,H + \sin^2(b)\,H) \\
&\qquad = (\sin(b)\,D + \cos(b)\,E)\,(\cos^2(b)\,F + \sin^2(b)\,G + 2\cos(b)\sin(b)\,H)
\end{aligned}
$$

=>

$$
\begin{aligned}
&(\cos^2(b)\sin(b)\,D\,F - \sin(b)\cos^2(b)\,D\,G - \cos^3(b)\,D\,H + \cos(b)\sin^2(b)\,D\,H) \\
&+ (\cos(b)\sin^2(b)\,E\,F - \sin^2(b)\cos(b)\,E\,G - \sin(b)\cos^2(b)\,E\,H + \sin^3(b)\,E\,H) \\
&\qquad = (\sin(b)\cos^2(b)\,D\,F + \sin^3(b)\,D\,G + 2\cos(b)\sin^2(b)\,D\,H) \\
&\qquad\qquad + (\cos^3(b)\,E\,F + \sin^2(b)\cos(b)\,E\,G + 2\cos^2(b)\sin(b)\,E\,H)
\end{aligned}
$$

=>

$$
\begin{aligned}
&\sin^3(b)\,(E\,H - D\,G) + \sin^2(b)\cos(b)\,(D\,H + E\,F - E\,G - 2\,D\,H - E\,G) \\
&+ \sin(b)\cos^2(b)\,(D\,F - D\,G - E\,H - D\,F - 2\,E\,H) - \cos^3(b)\,(D\,H + E\,F) = 0
\end{aligned}
$$

=>

$$
\begin{aligned}
g(b) &= \tan^3(b)\,(E\,H - D\,G) + \tan^2(b)\,(D\,H + E\,F - E\,G - 2\,D\,H - E\,G) \\
&\qquad + \tan(b)\,(D\,F - D\,G - E\,H - D\,F - 2\,E\,H) - (D\,H + E\,F) = 0 \\
&= \tan^3(b)\,(E\,H - D\,G) + \tan^2(b)\,(E\,F - D\,H - 2\,E\,G) \\
&\qquad - \tan(b)\,(D\,G + 3\,E\,H) - (D\,H + E\,F) = 0
\end{aligned}
$$

=>

$$
\begin{aligned}
g'(b) &= 3\tan^2(b)\sec^2(b)\,(E\,H - D\,G) + 2\tan(b)\sec^2(b)\,(E\,F - D\,H - 2\,E\,G) \\
&\qquad - \sec^2(b)\,(D\,G + 3\,E\,H)
\end{aligned}
$$

These are the equations needed for the Newton's Approximation method for determining the phase shift of the incoming pinger signal with respect to the internal reference signal.

This is the end of this White Paper. The use of these techniques without proper acknowledgement of the author in all written works will cause your sub to be cursed and sink to the bottom of the competition arena. Be sure to read the companion White Papers on the hydrophones mathematical model, Hydrophone Sampling Techniques, and Synchronous Fourier Transforms. Coming soon to a website near you!